Large Scale Structure Learning of Conditional Gaussian Graphical Models

Manzil Zaheer (manzil@cmu.edu)
Carlos Ramírez (carlosrc@cmu.edu)
Soumya Batra (soumyabatra@cmu.edu)

Advised by Mladen Kolar (mkolar@chicagobooth.edu)

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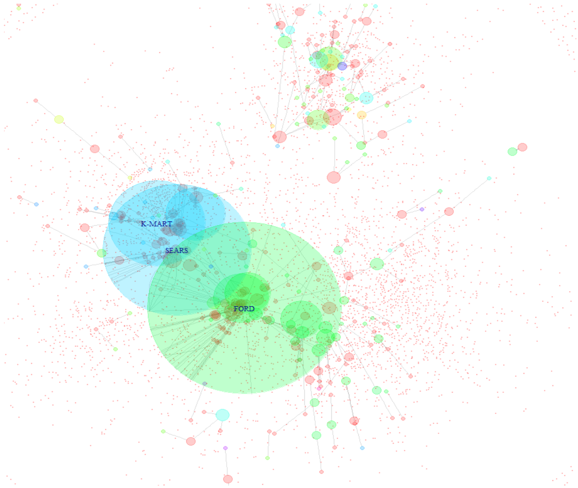
Motivation

- GGMs provide useful framework to represent relationships in complex systems.
- GGMs can be thought as undirected graphs of a set of random variables following a multivariate Gaussian distribution where
  - Nodes represents random variables
  - An edge between two nodes is absent if and only if the two r.v.s represented by those nodes are independent conditional on all other variables
- Modern financial markets are complex systems where uncovering relationships among firms may be of particular interest
  - Firms have become increasingly linked to each other through a complex and usually opaque network of relationships
  - Uncovering those relationships may be useful to predict future firm performance and thus firm prices and returns

Customer Supplier Network

Customer-Supplier Network Dynamics. Year: 1980



Colors represent major industry clusters (first two digits of SIC code)

Problem Abstraction

Assumption

$$X \mid Z = z \sim \mathcal{N}(\mu(z), \Sigma(z))$$

- Goal: Learn the conditional independence relationships among components of vector X given $Z = z$.
  - Let $\Omega(z) = \Sigma(z)^{-1} = (\omega_{ab}(z))_{a,b \in [p] \times [p]}$
  - Pattern of non-zero elements of this matrix encodes the conditional independencies

$$X_a \perp X_b \mid X_{-ab}, Z = z \quad \Leftrightarrow \quad \omega_{ab}(z) = 0$$

- Our data set consists of n time instances $\{z_1, \cdots, z_n\}$. At each z_i, we observe n_i instances of data vector x_{ij}

Optimization Problem

Optimization Problem

$$\min_{\Omega(\cdot)\in\mathcal{F}} \left\{ \sum_{i\in[n]} \left(\operatorname{tr}(C_i\Omega(z_i)) - \log|\Omega(z_i)| + \mu\|\Omega(z_i)\|_1\right) + \lambda \operatorname{pen}\left(\{\Omega(z_i)\}_{i\in[n]}\right) \right\}$$

- Desired properties for penalty function
  - Perform model selection and control the smoothness of the estimator $\omega_{ab}(z)$
  - Encourage the successive function values $\omega_{ab}(z_i), \omega_{ab}(z_{i+1})$ to be close
- Our choice of penalty function:

$$\operatorname{pen}\left(\{\Omega(z_i)\}_{i\in[n]}\right) = \sum_{i\in[n]} \left(\sqrt{\sum_{a,b} (\omega_{ab}(z_{i+1}) - \omega_{ab}(z_i))^2} \right)$$

Parallelizing

Original Optimization Problem

$$\min_{\Omega} \left\{ \sum_{i=1}^{n} (\mathrm{tr}(C_i\Omega_i - \log|\Omega_i| + \mu\|\Omega_i\|_1) + \lambda \sum_{i=1}^{n-1} \|\Omega_{i+1} - \Omega_i\|_F \right\}$$

Rewrite for parallelization

$$\min_{\Omega, R} \left\{ \sum_{i=1}^{n} (\mathrm{tr}(C_i\Omega_i) - \log|\Omega_i| + \mu\|\Omega_i\|_1) + \lambda \sum_{i=1}^{n-1} \|R_i\|_F \right\}$$

subject to: $R_i = \Omega_{i+1} - \Omega_i$

ADMM Form

Define constraint set $\mathcal{C} = \{(\Omega, R) : R_i = \Omega_{i+1} - \Omega_i\}$

$$\min_{\Omega, R, W, S} \left\{ \sum_{i=1}^{n} (\mathrm{tr}(C_i\Omega_i) - \log|\Omega_i| + \mu\|\Omega_i\|_1) + \lambda \sum_{i=1}^{n-1} \|R_i\|_F + I_{\mathcal{C}}(W, S) \right\}$$

subject to: $\Omega_i = W_i \quad R_i = S_i$

ADMM Steps

1. $2n-1$ Independent Optimizations

$$\Omega_i^{k+1} := \operatorname*{arg\,min}_{\Omega_i \succ 0} \left\{ \mathrm{tr}(C_i \Omega_i) - \log|\Omega_i| + \frac{\rho}{2}\|\Omega_i - W_i^k + U_i^k\|_2^2 \right\}$$

$$R_i^{k+1} := \operatorname*{arg\,min}_{R_i} \left\{ \lambda\|R_i\| + \frac{\rho}{2}\|R_i - S_i^k + T_i^k\|_2^2 \right\}$$

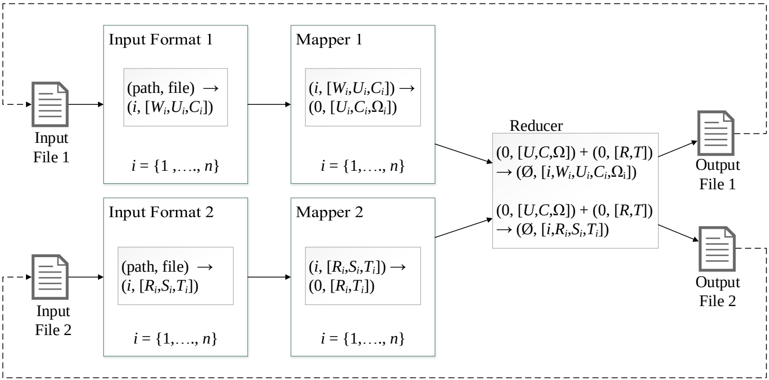
2. Projection onto the constraint set

$$(W, S) := \Pi_{\mathcal{C}}(\Omega^{k+1} + U^k, R^{k+1} + T^k)$$

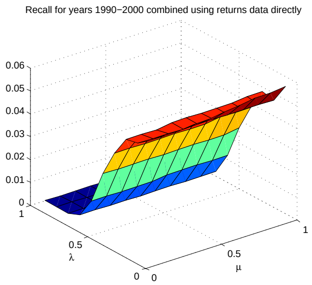
3. Scaled dual variables update

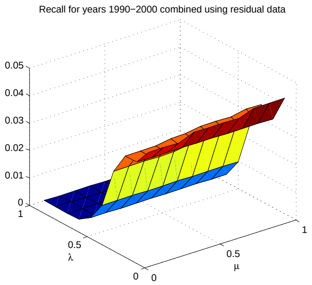
$$U_i^{k+1} := U_i^k + (\Omega_i^{k+1} - W_i^{k+1})$$
$$T_i^{k+1} := T_i^k + (R_i^{k+1} - S_i^{k+1})$$

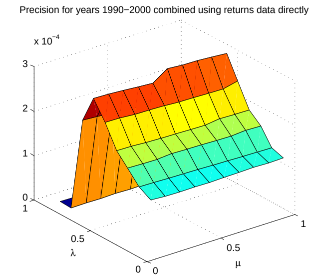
Distributed Implementation: Hadoop

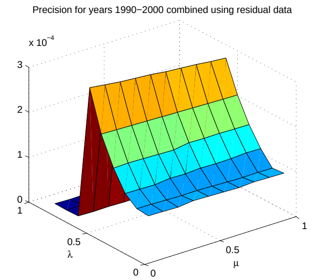


Estimation Quality ☹

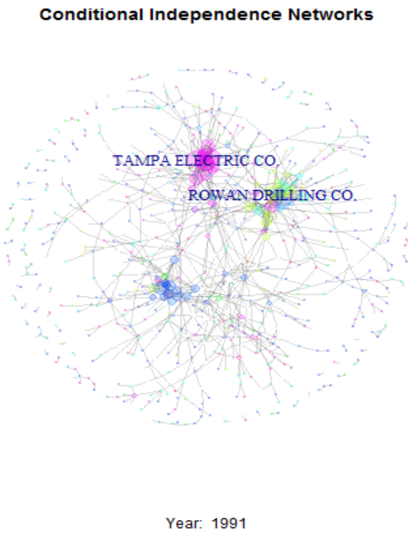








Estimated graph for 1991



Conclusions

- Conditional GGMs provide an important tool to uncover relationships among different variables in complex systems.
- Using returns data the use of graphical models seems to uncover industry relationships among firms
- However, conditional GGMs does not provide further information about the nature of such relationships (besides the identification of industry clusters).
  - We may need to include more information about firms to understand better the nature of such relationships
- Future Work
  - Handle missing data
  - Feature engineering

Thank You